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PLASTIC BENDING OF A STRIP FOR A YIELD CRITERION DEPENDING ON THE MEAN STRESS

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The plane-strain plastic bending of a wide strip is considered under the assumption that the material of the strip obeys the Coulomb–Mohr yield criterion and the two types of kinematic relations proposed by Spencer and Hill.

Key words: *bending, Coulomb–Mohr criterion, plasticity, large strains.*

The bending of a strip under large plane-strain conditions is considered in [1–6] using several models of a rigid-plastic body. In all these cases, the plastic flow is assumed to be independent of the mean stress. This assumption is justified for most metallic materials. However, models for materials with a yield criterion dependent on the mean stress have been developed for granular materials and soils, for which bending is not a typical mode of deformation. These models are reviewed in [7]. Nevertheless, the yield criteria for modern metallic materials, such as some aluminum alloys and steels, exhibit a dependence on the mean stress provided the incompressibility condition holds [8–11]. For these materials, the models of [7] are applicable and bending is an important mode of deformation, for example, during plastic metal working. In the present paper, a solution is constructed for the in-plane bending of a strip using the models proposed by Hill [1] and Spencer [12]. It is shown that these solutions coincide for this type of deformation. The solution extends the solution obtained in [1] to the model of an ideal rigid-plastic material. However, the new solution is based on the principally different approach considered in [13].

In the polar coordinates r and θ used below, the model proposed in [12] for a material under plain strain is defined by the equations

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \qquad \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} = 0; \tag{1}$$

$$(\sigma_{rr} + \sigma_{\theta\theta})\sin\varphi + [(\sigma_{rr} - \sigma_{\theta\theta})^2 + 4\sigma_{r\theta}^2]^{1/2} = 2k\cos\varphi; \tag{2}$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = 0; \tag{3}$$

$$\sin 2\psi\Big(\frac{\partial u}{\partial r} - \frac{u}{r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\Big) - \cos 2\psi\Big(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\Big) + \sin\varphi\Big(\frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial \theta} - 2\,\frac{d\psi}{dt}\Big) = 0. \tag{4}$$

Here (1) are the equilibrium equations, (2) is the Coulomb–Mohr yield criterion, (3) is the incompressibility equation, and (4) is the stress–strain relation; σ_{rr}, $\sigma_{\theta\theta}$, and $\sigma_{r\theta}$ are the stress-tensor components, u and v are the projections of the velocity vector onto the r and θ directions, respectively, ψ is the angle between the maximum principal stress and the r axis, which is reckoned counterclockwise from the axis, φ is the internal-friction angle, k is the adhesion coefficient, and d/dt denotes the total derivative with respect to time. In the model proposed in [1], Eq. (4) is replaced by the equation

$$\sin 2\psi\Big(\frac{\partial u}{\partial r} - \frac{u}{r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\Big) - \cos 2\psi\Big(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\Big) = 0. \tag{5}$$

The equations of classical plasticity are obtained for $\varphi = 0$. Here k is the shear yield point.

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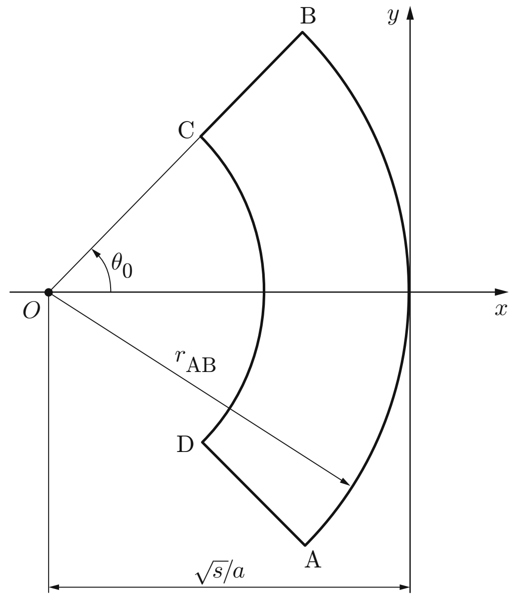

Fig. 1. Geometry of the process.

We denote the initial thickness and length of the strip by H_0 and $2L_0$, respectively. For pure bending, the final and all intermediate shapes of the region being deformed are determined by two circular arcs AB and CD and two straight lines AD and CB (Fig. 1). The process is symmetric about the x axis, and the y axis is tangent to the arc AB at each instant of time. Below, we confine our attention to the region $y \geqslant 0$. We denote the angle between the straight line CB and the x axis by θ_0 and the radii of the arcs AB and CD by $r_{\rm AB}$ and $r_{\rm CD}$, respectively. At the initial time ($\theta_0 = 0$), $r_{\rm AB} \to \infty$ and $r_{\rm CD} \to \infty$, the lengths of the arcs AB and CD are equal to $2L_0$, and the lengths of the straight lines CB and AD are equal to H_0. Aleksandrov and Dixon [13], introduced Lagrangian coordinates ζ and η that presumably coincide with the principal stress trajectories:

$$\frac{x}{H_0} = \sqrt{\frac{\zeta}{a} + \frac{s}{a^2}}\,\cos{(2a\eta)} - \frac{\sqrt{s}}{a}, \qquad \frac{y}{H_0} = \sqrt{\frac{\zeta}{a} + \frac{s}{a^2}}\,\sin{(2a\eta)}. \tag{6}$$

Here a is an arbitrary monotonically increasing function of time and s is an arbitrary function of a. Moreover, $a = 0$ for $t = 0$. It is shown in [13] that transformation (6) satisfies the incompressibility equation (3) for any function $s(a)$ and that at the initial moment, $x = \zeta H_0$ and $y = \eta H_0$ if

$$s = 1/4 \tag{7}$$

for $a = 0$. Thus, if relation (7) is satisfied, in the chosen coordinate system (x, y) (see Fig. 1) we have $\zeta = 0$ on AB, $\zeta = -1$ on CD, $\eta = 0$ on the symmetry axis, and $\eta = L_0/H_0$ on BC. For convenience, we introduce a polar coordinate system with origin at $x = -\sqrt{s}/a$, $y = 0$ that moves along the x axis (Fig. 1). Using (6), we obtain

$$\frac{r}{H_0} = \sqrt{\frac{\zeta}{a} + \frac{s}{a^2}}, \qquad \theta = 2a\eta. \tag{8}$$

From this, $r_{\rm AB}$, $r_{\rm CD}$, θ_0, and the current thickness h are expressed as

$$\frac{r_{\rm AB}}{H_0} = \frac{\sqrt{s}}{a}, \qquad \frac{r_{\rm CD}}{H_0} = \sqrt{\frac{s}{a^2} - \frac{1}{a}}, \qquad \theta_0 = \frac{2aL_0}{H_0}, \qquad \frac{h}{H_0} = \frac{\sqrt{s} - \sqrt{s-a}}{a}. \tag{9}$$

Thus, the function $s(a)$ completely defines the geometry of the deformed region. For the pure bending of a strip from rigid-plastic materials, a neutral line exists on which the stresses are discontinuous and the strain rates vanish. The strain rates can be determined from (6) by direct differentiation. In particular, the equivalent strain rate (the second invariant of the strain rate tensor) is given by

$$\xi_{\rm eq} = \frac{|\zeta + ds/da|}{\sqrt{3}\,(\zeta a + s)}\,\frac{da}{dt}. \tag{10}$$

Equation (10) defines the neutral line in Lagrangian coordinates

$$\zeta = \zeta_n = -\frac{ds}{da}. \tag{11}$$

From (8) it follows that at each time, the coordinate lines of the Lagrangian system (ζ, η) coincide with the coordinate lines of the system (r, θ). Thus, $\sigma_{r\theta} = 0$ at each point of the deformed region and, in particular, on its boundary, which is one of the boundary condition for pure bending of a strip. Moreover, the stresses σ_{rr} and $\sigma_{\theta\theta}$ are principal stresses such that $\sigma_{rr} < \sigma_{\theta\theta}$ (and $\psi = \pi/2$) in the range $\zeta_n < \zeta \leqslant 0$ and $\sigma_{rr} > \sigma_{\theta\theta}$ (and $\psi = 0$) in the range $-1 \leqslant \zeta < \zeta_n$. In this case, the yield criterion (2) becomes

$$(\sigma_{rr} + \sigma_{\theta\theta}) \sin\varphi \mp (\sigma_{rr} - \sigma_{\theta\theta}) = 2k \cos\varphi, \tag{12}$$

where the upper sign refers to the region $\zeta_n < \zeta \leqslant 0$ and the lower sign to the region $-1 \leqslant \zeta < \zeta_n$. The second equilibrium equation (1) and condition (12) show that the stresses are independent of θ. Using (8), we obtain

$$\frac{\partial}{\partial r} = \frac{\partial \zeta}{\partial r}\frac{\partial}{\partial \zeta} = \frac{2ar}{H_0^2}\frac{\partial}{\partial \zeta}. \tag{13}$$

From the first equilibrium equation (1), differentiating with respect to ζ using (13), eliminating r by means of (8), and taking into account that $\sigma_{r\theta} = 0$, we obtain

$$2(\zeta a + s)\frac{\partial \sigma_{rr}}{\partial \zeta} + a(\sigma_{rr} - \sigma_{\theta\theta}) = 0. \tag{14}$$

Eliminating $\sigma_{\theta\theta}$ from this equation by means of (12) and integrating over the regions $\zeta_n < \zeta \leqslant 0$ and $-1 \leqslant \zeta < \zeta_n$ for the boundary conditions $\sigma_{rr} = 0$ for $\zeta = 0$ and $\zeta = -1$, we obtain

$$\sigma_{rr} = k \cot\varphi[1 - s^{n_1}(\zeta a + s)^{-n_1}], \qquad n_1 = \sin\varphi/(1 + \sin\varphi) \tag{15}$$

in the region $\zeta_n < \zeta \leqslant 0$ and

$$\sigma_{rr} = k \cot\varphi[1 - (s - a)^{-n_2}(\zeta a + s)^{n_2}], \qquad n_2 = \sin\varphi/(1 - \sin\varphi) \tag{16}$$

in the region $-1 \leqslant \zeta < \zeta_n$. Since the stress σ_{rr} should be continuous for $\zeta = \zeta_n$ [ζ_n is determined by Eq. (11)], relations (15) and (16) imply the equation for s:

$$(s - a)^{-n_2}\Big(s - a\frac{ds}{da}\Big)^{n_2} = s^{n_1}\Big(s - a\frac{ds}{da}\Big)^{-n_1}.$$

This equation can be written as

$$a\frac{ds}{da} = s - s^{n_3}(s - a)^{n_4}, \qquad n_3 = \frac{\cos^2\varphi}{2(1 + \sin\varphi)}, \qquad n_4 = \frac{\cos^2\varphi}{2(1 - \sin\varphi)}. \tag{17}$$

Equation (17) should be solved numerically using condition (7). After that, all geometrical parameters of the process can be determined from (9). It is clear that Eq. (17) has a solution for $s \geqslant a$. The physical meaning of this inequality is that $r_{\rm CD} = 0$ for $s = a$, as follows from (9). We also note that the right and left sides of Eq. (17) vanish at the initial time. We determine the derivative ds/da for $a = 0$ using the condition that the principal vector of the external forces applied to the boundary segment CB (Fig. 1) vanishes. In the Cartesian coordinate system shown in Fig. 1, the stress σ_{xx} vanishes at each point at the initial time. Thus, from the yield criterion (12), in which σ_{rr} should be replaced by σ_{xx} and $\sigma_{\theta\theta}$ by σ_{yy}, we have

$$\sigma_{yy} = 2k\cos\varphi/(1 + \sin\varphi) \tag{18}$$

for $\zeta_n < \zeta \leqslant 0$ and

$$\sigma_{yy} = -2k\cos\varphi/(1 - \sin\varphi) \tag{19}$$

for $-1 \leqslant \zeta < \zeta_n$. Setting the principal vector of the external forces to zero, from Eqs. (18) and (19) we obtain

$$\frac{2k\cos\varphi}{1 + \sin\varphi}\zeta_n = \frac{2k\cos\varphi}{1 - \sin\varphi}(1 + \zeta_n)$$

or, with allowance for (11),

$$\frac{ds}{da} = -\zeta_n = \frac{1 + \sin\varphi}{2} \tag{20}$$

for $a = 0$. Condition (20) is used for numerical solution of Eq. (17).

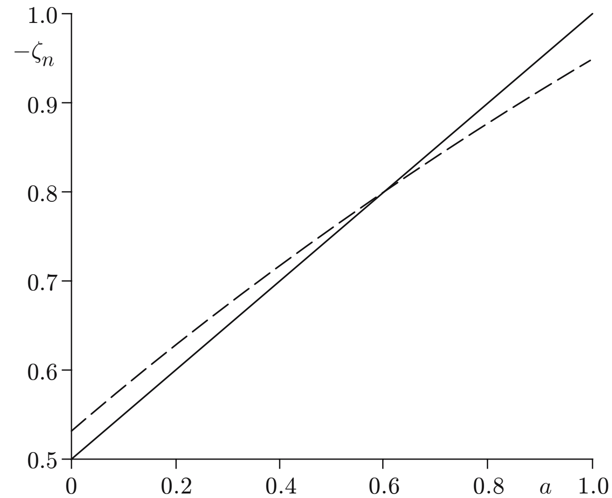

Fig. 2. Effect of the angle φ on the position of the neutral axis

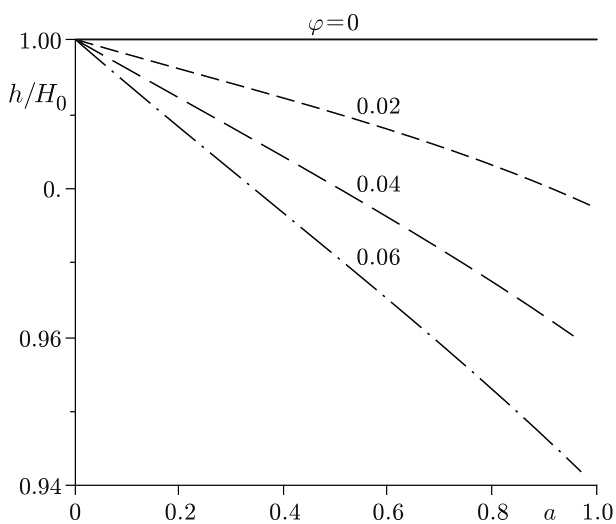

Fig. 3

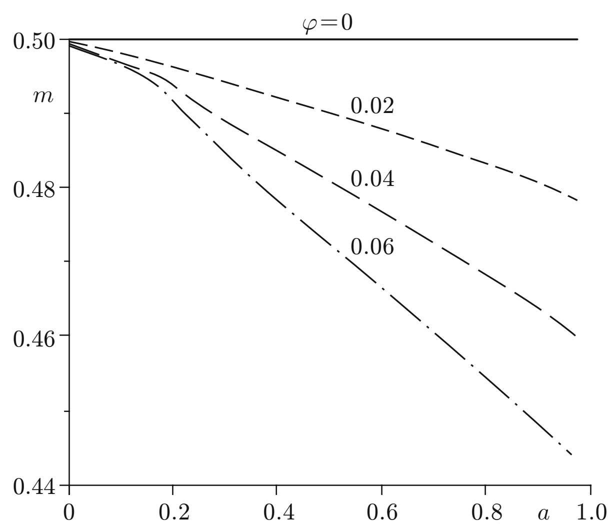

Fig. 4

Fig. 3. Effect of the angle φ on the current thickness of the strip

Fig. 4. Effect of the angle φ on the dimensionless bending moment

The bending moment is given by

$$M = \int_{r_{\rm CD}}^{r_{\rm AB}} \sigma_{\theta\theta} r \, dr. \tag{21}$$

Transforming (21) to the Lagrangian variable by means of (8), writing $\sigma_{\theta\theta}$ as a function of a and ζ with the help of (12), (15), and (16), and integrating, we obtain

$$m = \frac{M}{kH_0^2} = \frac{\cos\varphi}{2a^2}\Big[2s - a\cos\varphi + \frac{s^{n_1}(\zeta_n a + s)^{1-n_1}}{n_2} - \frac{(s-a)^{-n_2}(\zeta_n a + s)^{1+n_2}}{n_1}\Big]. \tag{22}$$

Here s and ζ_n are known functions of a due to the numerical solution of Eq. (17) and relation (11). For $a = 0$, the bending moment is defined by means of (18) and (19) as

$$m = \frac{M}{kH_0^2} = \frac{\cos\varphi}{2}. \tag{23}$$

It remains to show that Eqs. (4) and (5) are satisfied. Since the stresses and strain rates do not depend on θ, it suffices to examine the solution on the line $\theta = 0$. Due to of symmetry on this line, we have $v = 0$ and $\partial u/\partial\theta = 0$. Consequently, $\partial v/\partial r = 0$. In addition, $\psi = 0$ or $\psi = \pi/2$. Substitution of these relations into (4) and (5) shows that the equations are satisfied. Thus, in the case considered, the solutions for the two models coincide.

In [9], the quantity φ was determined experimentally for several steel grades. In the notation of the present paper, the interval of φ can be written as $0.014 < \varphi < 0.064$. For these materials, Fig. 2 shows the position of the neutral line versus the parameter a for two values of the angle φ (the solid curve refers to $\varphi = 0$ and the dashed curve to $\varphi = 0.064$). These curves were determined by solving Eq. (17) with the use of relation (11). It is interesting to note that in the initial stage of the process, the neutral line in materials whose yield criterion depends on the mean stress is closer to the concave surface CD (Fig. 1) than that in materials that obey the von Mises yield criterion and, after a certain level of strain, it is closer to the convex surface AB. The thickness of the strip determined from (9) and the dimensionless bending moment calculated from (22) and (23) are shown in Figs. 3 and 4, respectively, for various values of φ. For $\varphi = 0$, the curves in both figures correspond to the solution obtained by the classical theory of plasticity in [1]. We note that in accordance with (9), the quantity a, which is used in the figures as an independent variable, is proportional to the angle θ_0, which has a clear physical meaning. One can see from Fig. 4 that the bending moment decreases during the deformation of the strip (except for the solution obtained by the classical theory of plasticity).

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